Dissipation fluctuations of a passive scalar advected by a random velocity field

Victor Yakhot
Fluid Dynamics Research Center, Princeton University, Princeton, New Jersey, 08544-0710
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The dynamics of fluctuations of the dissipation rate of a passive scalar advected by a rapidly changing in time Gaussian random velocity field with the variance $\overline{[v_i(x)-v_i(x+r)]^2}\propto r^{\xi}$ is considered. It is shown that when $\xi/d\to 0$ the dissipation correlation function $\langle\mathcal{E}(x)\mathcal{E}(x+r)\rangle\propto r^{\gamma}$ with $\gamma=-4\xi/(d+2)$ in agreement with the recent works of Gawedzkii and Kupianen [Phys. Rev. Lett. **75**, 3608 (1995)], and Chertkov *et al.* [Phys. Rev. E **52**, 4924 (1995)]. It is shown that in this limit the fourth-order moment of the scalar difference is completely described in terms of the second-order moment and the scalar dissipation rate correlation function. [S1063-651X(96)11909-7]

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We consider the problem of a passive scalar $T(\boldsymbol{x},t)$ advection by a rapidly changing in time random velocity field obeying the following equation of motion [1]:

$$\partial_t T+\boldsymbol{v}\cdot\boldsymbol{\nabla}T=f+\nu\nabla^2 T. \tag{1}$$

The rapidly changing in time incompressible $(\boldsymbol{\nabla}\cdot\boldsymbol{v}=0)$ random velocity field is defined by the correlation function

$$\langle v_i(\boldsymbol{x},t)v_j(\boldsymbol{x}',t')\rangle=\delta(t-t')D_{ij}(\boldsymbol{x}-\boldsymbol{x}')\equiv\delta(t-t')[D_{ij}(0)-S_{ij}(\boldsymbol{r})]. \tag{2}$$

The structure function $S_{ij}(r)$ is defined as

$$S_{ij}(\mathbf{r})=Dr^{\xi}[(d+\xi-1)\delta_{ij}-\xi n_i n_j], \tag{3}$$

where d is the dimension of space, $0\leqslant\xi\leqslant 2$ and $n_i=r_i/r$.

The force f is Gaussian, rapidly changing in time and isotropic

$$\langle f(\mathbf{0},t)f(\boldsymbol{r},t')\rangle=\delta(t-t')F(r) \tag{4}$$

and the forcing functions act at the large scales $r\sim L$ only. This means that when $r/L\to 0$, setting all amplitudes equal to unity, the functions $F(r)=1+O[(r/L)^2]$. The infrared cutoff is denoted as L and the ultraviolet cutoff (Kolmogorov scale) is $r_d=(\nu/D)^{1/\xi}\to 0$.

Evaluation of the energy spectrum of velocity fluctuations in turbulent flow remains one of the last important problems of classical physics. At the present time no argument which is better founded than the heuristic Kolmogorov 1941 theory (K41), produced more than half a century ago, exist. Although the predictions of K41 have been confirmed by experimental data with remarkable accuracy, one cannot argue with certainty that no correction to the 5/3-energy spectrum exist. Measurements of the higher-order moments velocity difference revealed the so-called anomalous scaling, i.e., $s_n=\langle[u(\mathbf{x})-u(\mathbf{x}+\mathbf{r})]^{2n}\rangle\propto r^{\xi_n}$ with the exponents $\xi_n<n\xi_2$. Understanding the origin of anomalous scaling of the high-order moments s_n still remains one of the most challenging tasks of turbulence theory.

The problem defined by (1)-(3) is very interesting since it removes the trouble of evaluation of the scalar variance $S_2=\langle[T(\mathbf{x})-T(\mathbf{x}+\mathbf{r})]^2\rangle\propto r^{\xi_2}$ and, as a consequence, enables one to concentrate on investigation of the scaling properties of the higher-order moments. Indeed, it easy to show [1] that in the limit $\nu\to 0$ and for all $r\gg r_d$

$$\frac{\partial S_2}{\partial t}-\frac{2}{r^{d-1}}\frac{\partial}{\partial r}\left(Dr^{d-1+\xi}\frac{\partial S_2}{\partial r}\right)=-\overline{\mathcal{E}}, \tag{5}$$

where the scalar dissipation rate $\mathcal{E}=\nu\langle(\partial_{xi}T)^2\rangle=O(1)$. The first term on the right side of (5) defines the so called turbulent diffusivity describing the scalar transfer by the random velocity field. The contribution

$$\nu\frac{\partial^2 S_2(r)}{\partial r^2}$$

is equal to zero in the inertial range when $\nu\to 0$ and $r\gg r_d$. This is the reason why all $O[\nu\phi(r)]$ terms are neglected in what follows. However, when $r\to r_d$ one has to be careful since, for example, $\nu S''(r)\to\langle\mathcal{E}\rangle=O(1)$. We can see that the solution of problem (5) is $S_2\propto r^{\xi_2}$ with

$$\xi_2=2-\xi.$$

This (''normal'') scaling is a reflection of the fact that due to the scalar variance conservation law the scalar variance flux $\langle\mathcal{E}\rangle$ is the only parameter characterizing the dynamics at the scales $r_d\ll r\ll L$. Indeed, the Fourier transform of right side of (5), which is related to sources and sinks at the intermediate scales, is equal to zero for all $r\ll L$. The effective-diffusivity term in (5) is typical of all problems of scalar or vector advection by the rapidly changing in time random velocity field. However, in the case of nonconserved quantities the right side of (5) can also involve nonanalytic contributions, proportional to $S_2 r^{\xi-2}$ which, in principle can introduce anomalous scaling, dominated by the solutions of the corresponding homogeneous equation. In this case the scaling exponent, which cannot be determined on dimensional grounds, is derived from analysis of zero modes, reflecting the infrared properties of the system. An excellent illustration of these ideas was recently worked out by Vergassola [2] who, considering the problem of magnetic field $\mathbf{H}$ advection by a random velocity field, calculated the correlation function $\langle H_i(x)H_i(x+r)\rangle\propto r^{\xi_h}$ with the anomalous exponent

ξ_h dominated by the zero modes. It was pointed out by Borue and Yakhot [3] that in the same system the fluctuations of helicity, which unlike a magnetic field is an inviscid invariant, are characterized by the normal scaling, derived from dimensional considerations with the helicity flux as a governing parameter.

The first crack in the tough problem of anomalous scaling of the higher-order moments in statistical hydrodynamics was produced recently in two important papers by Gawedzkii and Kupianen [4] and Chertkov *et al.* [5] who, investigating the scaling of the fourth-order moment $S_4 \propto r^{\xi_4}$ in the vicinity of the Gaussian limits $\xi \to 0$ [4] and $d \to \infty$ [5], obtained

$$\xi_4 = 2\xi_2 - \frac{4\xi}{d+2}. \tag{6}$$

This relation manifests a theoretical breakthrough since for the first time it demonstrates an anomalous scaling of a higher-order moment in a problem related to turbulence and turbulent transfer.

The authors of Refs. [4] and [5] identified the reason for the anomalous scaling of S_4 as originating from nontrivial scaling exponent of the dissipation rate fluctuations. It is easy to show that when $r \to 0$ $(r \gg r_d)$

$$\frac{\partial S_4}{\partial t} \approx \frac{2}{r^{d-1}} \frac{\partial}{\partial r}\left(D r^{d-1+\xi} \frac{\partial S_4}{\partial r}\right) + b\langle \mathcal{E}(\mathbf{x})\mathcal{E}(\mathbf{x}+\mathbf{r})\rangle r^{\xi_2}, \tag{7}$$

where the value of the coefficient b is easily derived from the theory. This equation is derived in the following way. We have from (1)

$$\frac{\partial S_4}{\partial t} \approx \frac{2}{r^{d-1}} \frac{\partial}{\partial r}\left(D r^{d-1+\xi} \frac{\partial S_4}{\partial r}\right) - G_{\epsilon K},$$

where

$$G_{\epsilon K} = -12\langle [\mathcal{E}(1)+\mathcal{E}(2)][T(2)-T(1)]^2\rangle.$$

The principle contribution to the equation for $G_{\epsilon K}$ is

$$\frac{\partial G_{\epsilon K}}{\partial t} \approx \frac{2}{r^{d-1}} \frac{\partial}{\partial r}\left(D r^{d-1+\xi} \frac{\partial G_{\epsilon K}}{\partial r}\right) + \langle \mathcal{E}(1)\mathcal{E}(2)\rangle$$

which in the statistically steady state gives Eq. (7). We can see that

$$S_4 \propto r^{2\xi_2+\gamma} \tag{8}$$

and the exponent γ characterizes the dissipation rate fluctuations

$$\langle \mathcal{E}(x)\mathcal{E}(x+r)\rangle \propto r^{\gamma}. \tag{9}$$

Thus evaluation of the scaling exponent of $S_4(r)$ is equivalent to the determination of the dimensionality of the scalar dissipation rate which is the subject of this work. The goal of the paper is twofold. First, we rederive the important results of Refs. [4,5] using a different, though equivalent, procedure. Second, we attempt here to understand what can happen if parameter of the problem ξ/d is not too small. It will be shown that the most probable scenario is a crossover from the anomalous scaling of Refs. [4,5] to the normal scaling. According to the theory, presented here, the transition happens at $\xi/d \approx d/(5d-6)$, which introduces a numerical smallness justifying approximations involved in the derivation of this result.

Let us define the dissipation matrix

$$\mathcal{E}_{ij}(\mathbf{x},t) = \nu \frac{\partial T(x,t)}{\partial x_i} \frac{\partial T(x,t)}{\partial x_j} \tag{10}$$

and the dissipation correlation tensor

$$\Gamma_{ijkl}(x,x') = \mathcal{E}_{ij}(x)\mathcal{E}_{kl}(x'). \tag{11}$$

The trace of $\mathcal{E} \equiv \mathcal{E}_{ii}$ satisfies the equation

$$\partial_t \mathcal{E} + \boldsymbol{v}\cdot\boldsymbol{\Delta}^* \mathcal{E} = -2\partial_j v_i \mathcal{E}_{ij} - 2\nu^2(\partial_i\partial_j T)^2 + \nu\Delta\mathcal{E}. \tag{12}$$

We will be interested in the two-point correlation function of the scalar dissipation

$$G(r,t) = \langle \mathcal{E}(\mathbf{0},t)\mathcal{E}(\boldsymbol{r},t)\rangle. \tag{13}$$

For the rapidly changing in time Gaussian velocity field the equation of motion for $G(r,t)$ in the limit of zero viscosity has the form

$$\begin{aligned}\partial_t G = {} & \langle v_i v_j\rangle \partial_i\partial_j G + 4\langle v_k \partial_i v_j\rangle \partial_k \langle \mathcal{E}\mathcal{E}_{ij}\rangle + 4\langle \partial_i v_j \partial_a v_b\rangle \\ & \times \langle \mathcal{E}_{ij}\mathcal{E}_{ab}\rangle + 2\xi a G r_d^{\xi-2} - 2\langle Y(1)\mathcal{E}(2)\rangle \\ & - 2\langle Y(2)\mathcal{E}(1)\rangle\end{aligned} \tag{14}$$

where

$$Y = \nu^2(\partial_i\partial_j T)^2$$

is the local value of the dissipation rate of the scalar variance dissipation. The uv divergent $O(r_d^{\xi-2})$ contribution to (14) comes from

$$\left\langle \frac{\partial v_i(x)}{\partial x_j} \frac{\partial v_i(x)}{\partial x_j} \right\rangle$$

which is a single-point mean quantity understood as a limit of the two-point correlation function when the separation $|x-x'| \to r_d$. The $O(1)$ coefficient a is easily derived from definition (3). It is important that, as one can see from (3), that this term is proportional to ξ.

The equations of motion for the correlation function G is not closed due to the appearence of the $\langle Y\mathcal{E}\rangle$ and $\gamma_{ijkl} = \langle \Gamma_{ijkl}\rangle$ correlators on the right side of (14). Indeed, we are dealing here with an infinite chain of coupled equations generated by the dissipation term in (12). The equation for Y is

$$\begin{aligned}\partial_t Y + \boldsymbol{v}\cdot\boldsymbol{\Delta}^* Y = {} & -2\nu^2 \partial_j v_m Y_{miij} - 2\nu^2 \partial_i v_m Y_{mjji} \\ & -2\nu^2 \partial_i\partial_j v_m \partial_m T \partial_i\partial_j T + 2\nu^2 \partial_i\partial_j f \partial_i\partial_j T \\ & + 2\nu^3(\partial_i\partial_j T)(\partial_i\partial_j \Delta T),\end{aligned} \tag{15}$$

where

$$Y_{miij}=\nu^3(\partial_m\partial_i T)(\partial_i\partial_j T).$$

Equation (15) defines the dissipation of the scalar dissipation rate given by

$$Y_2=-4\nu^3(\nabla^3 T)^2, \tag{16}$$

where $(\nabla^n T)^2\equiv(\partial_{i1}\cdots\partial_{in}T)^2$. The equation for the correlation function $G_2=\langle Y_2(1)Y_2(2)\rangle$ involves the term

$$2G_2 r_d^{\xi-2}-2\langle Y_2(1)Y_3(2)\rangle-2\langle Y_2(2)Y_3(1)\rangle. \tag{17}$$

Then, we have to write the equations for the entire series of the dissipation rates

$$Y_n=-2^{n-1}\nu^n(\nabla^n T)^2 \tag{18}$$

with $n\geqslant 1$. The most important feature of all equations for $Y_n(1)Y_n(2)$ correlation functions is that it involves the following combinations:

$$2G_n r_d^{\xi-2}-2\langle Y_n(1)Y_{n+1}(2)\rangle-2\langle Y_n(2)Y_{n+1}(1)\rangle, \tag{19}$$

where $G_n=\langle Y_n(1)Y_n(2)\rangle$. We will see below that in all these equations the uv divergent terms cancel each other and the remaining $O(1)$ in the displacement r contributions are $O(d^{-n})$ which are small in the limit of large d. This conclusion agrees with the result of Refs. [4] and [5] where the evaluation of the anomalous scaling exponent of the fourth-order structure function $S_4(r)$ was conducted. It was shown there that the scaling is completely dominated by the kinetic energy dissipation anomaly which disappears in the limit $\xi=0$ and $d=\infty$. Thus, in this limit the effects, dominated by the fluctuations of $Y_n(x,t)$ with $n>1$, are small. It will be shown below that $2Gr_d^{\xi-2}-4\langle\mathcal{E}Y\rangle\approx(\xi/d)^2 Gr^{\xi-2}$ and for the time being, we can neglect both the $Y\mathcal{E}$ correlation functions and $O(r_d^{\xi-2})$ contributions to the equation for $G(r)$

$$\partial_t G=\langle v_i v_j\rangle\partial_i\partial_j G+4\langle v_k\partial_i v_j\rangle\partial_k\langle\mathcal{E}\mathcal{E}_{ij}\rangle$$
$$+4\langle\partial_i v_j\partial_a v_b\rangle\langle\mathcal{E}_{ij}\mathcal{E}_{ab}\rangle. \tag{20}$$

We will also need the equation for γ_{ijkl} which in the same approximation can be written as

$$\partial_t\gamma_{ijkl}=\langle v_\alpha v_\beta\rangle\partial_\alpha\partial_\beta\gamma_{ijkl}+4\langle v_\alpha\partial_j v_\beta\rangle\partial_\alpha\gamma_{i\beta kl}$$
$$+4\langle\partial_j v_\beta\partial_k v_\alpha\rangle\gamma_{i\beta\alpha l}. \tag{21}$$

It is assumed that in the limit $\nu\to 0$ the mean scalar dissipation rate is equal to the power of the external source $\langle\mathcal{E}\rangle\to 1$. Thus we have to consider the limiting procedure

$$\mathcal{E}=\lim_{\nu\to 0}\nu\lim_{x'\to x}\partial_{x'}\partial_x T(x')T(x). \tag{22}$$

Taking the mean we have

$$\langle\mathcal{E}\rangle=\lim_{\nu\to 0}\nu\lim_{x'\to x}\partial_r\partial_r\langle T(x')T(x)\rangle \tag{23}$$

evaluated at $r=r_d$. Since $\langle T(x)T(x')\rangle\propto r^{2-\xi}$ the mean dissipation rate is equal to unity. Thus the dissipation rate can be evaluated using the inertial range scalar correlation function at the separation distance $r=r_d$. Let us discuss the properties of the dissipation tensor (10) which can be defined as

$$\mathcal{E}_{ij}(\mathbf{x},\mathbf{x}',t)=\lim_{\nu\to 0}\nu\partial_{x_i'}T(x',t)\partial_{x_j}T(x,t) \tag{24}$$

when $|x-x'|\to r_d\to 0$.

We can see from (24) that evaluation of the operator $\mathcal{E}_{ij}$ is equivalent to calculation of the four-point correlation function in the limit of the two of the distances going to $r_d\to 0$. To calculate the dissipation correlation function one has to set two of the distances equal to r_d and average over all angles. This was the approach pursued in Refs. [4] and [5]. In a more familiar language, we are interested in the role of the scalar fluctuations at the scales $l\leqslant r_d$ in the inertial range dynamics of the fluctuations at the scales $r>>r_d$. To investigate this question and elucidate the details of the limiting procedure, we can average Eq. (1) over velocity fluctuations $v(\mathbf{k})$ with the wave numbers $k_d<k$ and derive the equation of motion for $\Theta(\mathbf{k})=T(\mathbf{k})$ for $k<<k_d$ and $\Theta(\mathbf{k})=0$ for $k>k_d$. The procedure involves writing the Fourier transform of (1)

$$T(\mathbf{k},\omega)=fg^o(\mathbf{k},\omega)+ig^o k_i(\mathbf{k},\omega)\int v_i(\mathbf{q},\Omega)$$
$$\times T(\mathbf{k}-\mathbf{q},\omega-\Omega)d\mathbf{q}\;d\Omega, \tag{25}$$

where the bare propagator is

$$g^o=(-i\omega+\nu k^2)^{-1}. \tag{26}$$

The correlation function of the velocity field can be written as

$$\langle v_\alpha(\mathbf{k})v_\beta(\mathbf{k}')\rangle\propto\Gamma\left(\frac{\xi+d}{2}\right)\Gamma\left(\frac{\xi+2}{2}\right)\sin\left(\frac{\pi\xi}{2}\right)$$
$$\times k^{-d-\xi}\left(\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}\right)\delta(\mathbf{k}+\mathbf{k}')\,\delta(t-t'). \tag{27}$$

When $\xi\ll 1$ this expression is $O(\xi)$. Eliminating the modes from the interval $k_d<k$ is done readily in the limit $d\to\infty$. It is easy to see by analyzing the diagrammatic expansion of the propagator that the one-loop correction, which is exact in the case of the scalar advected by the rapidly changing in time random velocity field, is $O(d-1/d\,\xi)$. Thus the sole effect of the small-scale modes on the scalar fluctuations $T(k)$ with $k\ll k_d$ is in the renormalization of diffusivity coefficient which becomes

$$\nu_1\approx\nu+O(\xi/d). \tag{28}$$

Since evaluating $\mathcal{E}_{ij}$ we are interested in the role of the small scales $l\approx r_d$ for the fixed separation distance r, the scale separation justifying the eddy diffusivity concept is not a problem here. Relation (28) shows that in the limit $d\to\infty$ the correction to the diffusivity coming from the modes $v(k)$ with $k_d<k$ tends to zero and the equation of motion for $\Theta(\mathbf{k},t)$ is exactly the same as (1) but with ν_1 instead of ν. This result is important since it tells us that setting two of the

separation distances ρ_{ij} and ρ_{mn} in the general n-point correlation function equal to r_d and averaging over all directions of the vectors $\boldsymbol{\rho}_{ij}$ and $\boldsymbol{\rho}_{mn}$ gives an $O(1/d)$ factor. As a consequence, we can assume the form of the tensor $\gamma_{ijkl}(r)$

$$\begin{aligned}\gamma_{ijkl}&=\langle \mathcal{E}_{ij}(\mathbf{x},\mathbf{x}',t)\mathcal{E}_{kl}(\mathbf{y}+\mathbf{r},\mathbf{y}'+\mathbf{r},t)\rangle\\&=\nu^2\langle \partial_{x_i'}T(x',t)\partial_{x_j}T(x,t)\partial_{y'+r_k}T(\mathbf{y}'+\mathbf{r},t)\\&\quad\times\partial_{y+r_l}T(y+r,t)\rangle=\partial_i\partial_j\partial_k\partial_l\varphi(r)+O(1/d\varphi''(r))\end{aligned}\tag{29}$$

in the limit $|\mathbf{x}-\mathbf{x}'|\to r_d$, $|\mathbf{y}-\mathbf{y}'|\to r_d$ and $r\gg r_d$. The yet unknown function φ depends only on the large separation distance r and must be determined from the dynamics of the system. The functional form of the second term on the right side of (29) will be derived below. The function $\varphi(r)$ is a mean of composite operators of the kind $\langle A(x)B(x+r)\rangle$ where $A(x)$ and $B(x+r)$ are the functions of $T(x)$ and $T(x+r)$, respectively, which can be represented in terms of the corresponding multipoint correlation functions with part of the separation distances equal to r_d. Thus expression (29) introduces the operator product expansion. We can write (29) in terms of the corresponding eight-point correlation functions with part of the separation distances equal to the dissipation scale r_d. This, however, is unneccessary since, due to the angular integrations the derivatives over $\rho_1=\mathbf{x}-\mathbf{x}'$ and $\rho_2=\mathbf{y}-\mathbf{y}'$ are small and can be neglected in the evaluation of the dissipation correlation function, when $d\to\infty$. Relation (29) also tells us that in the limit $d\to\infty$ the correlation function of the dissipation matrix is given by a potential tensor up to the $O(1/d^2)$ correction. This fact simplifies the calculations presented below and agrees with the conclusion of Refs. [4,5] based on the direct analysis of the four-point correlation functions. Evaluation of the operator product expansion (29) in a general case, not based on the potentiality property of γ_{ijkl}, is an extremely complex, unsolved, problem.

To demonstrate that in the limit $d\to\infty$ the equation for $\gamma_{ijkl}(r)$ preserves the potentiality property, we have to substitute (29) into (21), take $curl$ over one of the indices and prove that

$$\epsilon_{abi}\partial_b\frac{\partial\gamma_{ijkl}}{\partial t}=0.$$

The same relation must hold also if we take $curl$ over any other component j, k, or l. The calculation is very easy in the Fourier space where

$$\gamma_{ijkl}=k_ik_jk_kk_l\varphi(k)$$

and $\varphi(k)$ is the Fourier transform of $\varphi(r)$. The Fourier transform of the first term on the right side of (21) with the velocity correlation function (27) is

$$\begin{aligned}R_1=k_\alpha k_\beta\int d\mathbf{q}\ q^{-d-\xi}\left(\delta_{\alpha_\beta}-\frac{q_\alpha q_\beta}{q^2}\right)(k-q)_i\\\times(k-q)_l(k-q)_k(k-q)_j\varphi(|\mathbf{k}-\mathbf{q}|).\end{aligned}\tag{30}$$

It will be shown below that $\varphi(k)\propto k^{-\gamma-d-4}$ when $k>>1/L$ with $\gamma<0$ and $|\gamma|=O[\xi/(d+2)]$. Thus, the integrand in (30) is

$$O(q^{-\xi-d}|\mathbf{k}-\mathbf{q}|^{-\gamma-d}).$$

Since $\xi>0$, expression (30) is an infrared divergent integral in the limit $q\to 0$ while the limit $\mathbf{q}\to\mathbf{k}$ does not pose any problem. We see that $R_1=O(L^\xi)$. It is clear, however that this is an artifact of the Fourier space integration using the expression for the velocity correlation function (27). In fact, the principle contribution to the integral (30) comes from the interval $1/L\ll ak<q$ with $a\ll 1$, where the value of the $O(1)$ parameter a is not needed for the conclusions derived below. Since the integral is dominated by the infrared region, we set $\varphi(|\mathbf{k}-\mathbf{q}|)\approx\varphi(k)$ and evaluate (30) with the result

$$R_1\propto k^{2-\xi}\frac{d-1}{da^\xi\xi}k_ik_jk_kk_l\varphi(k)+O\left(\frac{k^{6-\xi}a^{2-\xi}}{(2-\xi)(d+2)}\varphi(k)\right).\tag{31}$$

Thus direct calculation shows that in the limit $d\to\infty$ the dissipation tensor correlation function is a potential tensor up to $O(1/d)$ corrections. This result is extremely important for what follows since it allows straightforward and simple evaluation of the scalar dissipation rate correlation function $G(r)$. It is interesting that evaluating Tr_{jk} of the right side of (31) (RS) gives

$$Tr_{jk}RS\propto k_ik_l,$$

which is a potential tensor in since the $curl$ over the remaining indices i and k is equal to zero.

The most general form of potential tensor γ_{ijkl} can be derived from an expression given in [6]

$$\begin{aligned}\gamma_{ijkl}=&A(r)n_in_jn_kn_l+C(r)(\delta_{ij}\delta_{kl}+\delta_{jl}\delta_{ik}+\delta_{jk}\delta_{il})\\&+B(r)(\delta_{ij}n_kn_l+\delta_{ik}n_ln_j+\delta_{il}n_jn_k+\delta_{kl}n_in_j\\&+\delta_{jl}n_in_k+\delta_{jk}n_in_l).\end{aligned}\tag{32}$$

The potentiality property simplifies the derivation enormously. First of all, the most general form of the fourth-order tensor involves, in addition to $A(r)$, $B(r)$, and $C(r)$ given above, two more which are in a potential case equal to zero. To calculate $\gamma_{ijkl}=\partial_i\partial_j\partial_a\partial_b\varphi(r)$ all we have to do is evaluate the fourth-order derivative of a scalar function and, comparing the resulting expression with (32) obtain the desired relations between $A(r)$, $B(r)$, and $C(r)$. The simple but bulky calculation gives

$$C=\frac{1}{r}\frac{\partial}{\partial r}\frac{\varphi'(r)}{r},$$

$$B(r)=rC'(r),$$

$$A(r)=r^3\frac{\partial}{\partial r}\frac{C'}{r}.$$

The trace of the dissipation tensor is equal to

$$G=\langle\mathcal{E}(x)\mathcal{E}(x+r)\rangle=A+2B(d+2)+C(d^2+2d)=r^2C''+rC'(2d+3)+d(d+2)C.$$

This expression can be rewritten as

$$G=\Delta\Delta\varphi=\frac{1}{r^{d-1}}\frac{\partial}{\partial r}r^{d-1}\frac{\partial}{\partial r}\left\{\frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left[r^{d-1}\frac{\partial\varphi(r)}{\partial r}\right]\right\}.$$

Substituting these definitions into (21) we have

$$\frac{\partial G}{\partial r}+4\xi\frac{\partial}{\partial r}[rC'+(d+2)C]+\frac{4\xi(d+\xi)}{r}C+4\xi(\xi-1)C'=0.$$

Seeking the solution in the form $C\propto r^\gamma$ so that

$$G\propto(d+2+\gamma)(d+\gamma)r^x,$$

we derive a cubic equation for the exponent γ

$$\gamma(d+\gamma+4\xi)(d+2+\gamma)+4\xi[d+\gamma(\xi-1)+\xi]=0, \tag{33}$$

which can be solved numerically in the interval $2<d<\infty$ and $0<\xi<2$. It can be seen readily that when $d\to\infty$ this equation gives

$$\gamma=-\frac{4\xi(d+\xi)}{(d+4\xi)(d+2)}\to-\frac{4\xi}{d+2} \tag{34}$$

which is exactly the result of Refs. [4] and [5].

Now we have to evaluate the correlation function $g(r)=2\langle Y(1)\mathcal{E}(2)\rangle-Gr_d^{\xi-2}$ neglected in the derivation of the above relations. In the vicinity of the Gaussian limit this calculation is not difficult and is easily done in the Fourier space. First of all, let us show that the the $O(Gr_d^{\xi-2})$ uv divergent term is canceled by the contributions coming from the $Y\mathcal{E}$ correlation function. The Fourier transform of the last two terms on the right side of (14) is

$$\nu^3\int\mathbf{dQ}\;\mathbf{dQ'dq}[\mathbf{Q}\cdot(\mathbf{q}-\mathbf{Q})]^2[\mathbf{Q'}\cdot(\mathbf{k}-\mathbf{q}-\mathbf{Q'})]T(\mathbf{Q})T(\mathbf{Q'})T(\mathbf{q}-\mathbf{Q})T(\mathbf{k}-\mathbf{q}-\mathbf{Q'})$$
$$\approx-\nu^3\int\mathbf{dQ}\;\mathbf{dq}[\mathbf{Q}\cdot(\mathbf{q}-\mathbf{Q})]^2[\mathbf{Q}\cdot(\mathbf{k}-\mathbf{q}+\mathbf{Q})]|T(Q)|^2|T(k-q+Q|^2\approx\nu^3k_d^{2+2\xi}\approx r_d^{\xi-2},$$

where the relation $\nu\propto r_d^\xi$ has been used. Since all other contributions to (14) are $O(1)$ the cancellation of the uv $O(r_d^{\xi-2})$ divergences, coming from the last three terms on the right side of (14), is an exact consequence of Eq. (14). Thus, the estimate showing that $\langle Y\mathcal{E}\rangle=O(r_d^{\xi-2})$, produced above, is proof of the uv divergences cancellation in (14). To evaluate the subleading terms we write

$$\nu^3\int\mathbf{dQ}\;\mathbf{dq}\;Q^2Q_iQ_mq_iq_m|T(Q)|^2|T(k-q-Q)|^2=O(1/d^2)$$

which is small in the limit $d\to\infty$ we are interested in.

It is possible to show that

$$2\xi aGr_d^{\xi-2}-2\langle Y(1)\mathcal{E}(2)\rangle-2\langle\mathcal{E}(1)Y(2)\rangle\approx-\frac{24\xi^2(d-1)}{d^2(d+2)}\langle\mathcal{E}(1)\mathcal{E}(2)\rangle r^{\xi-2}+O(\overline{\mathcal{E}^2}r^{\xi-2}). \tag{35}$$

Substituting this into (14) and (33) we see that in the limit of small ξ/d the neglected term is $O[(\xi/d)^2]$. In the same limit the contribution proportional to

$$\overline{\mathcal{E}^2}r^{\xi-2}=O(r^{\xi-2}) \tag{36}$$

is negligibly small when $r\to0$. Relation (35) has a simple physical meaning understood within a framework of derivation of the so called $K-\mathcal{E}$ model widely used for engineering computations of turbulent flows. The correlation function

$$\langle Y(1)\mathcal{E}(2)\rangle=O[\nu^3\partial^4T(1)\partial^2T(2)]$$

is a sum of the uv divergent $O(\nu/r_d^2\approx r_d^{\xi-2})$ contribution and the $O(1)=\nu_T/r^2$ term which is obtained by substituting one of the three ν factors by the ''turbulent diffusivity'' $\nu_T\approx r^\xi$. The factor 24 in the above relations is the result of diagrammatic expansion of the $\langle Y\mathcal{E}\rangle$ correlation function leading to the substitution of one of the ν's by ν_T. The factor $1/d(d+2)$ comes from the angular integration. The divergent contributions to Eq. (14) cancel each other and the remaining term has a structure presented above.

The most remarkable feature of formula (35) is that the $O(r^{\xi-2})$ term (36) can produce an extremely important physical effect when ξ/d is not too small. To illustrate the qualitative aspects of the phenomenon, we substitute (35) into (33) which is, strictly speaking, valid only when $\xi/d\to0$ and investigate the resulting equation for the scaling exponents. In this case the equation is

$$\gamma(d+\gamma+4\xi)(d+2+\gamma)+4\xi[d+\gamma(\xi-1)+\xi] \\ -\frac{24\xi^2(d-1)}{d}\frac{(d+2+\gamma)(d+\gamma)}{d(d+2)}+O\left[\overline{\mathcal{E}^2}\left(\frac{r}{L}\right)^{-\gamma}\right]=0. \quad (37)$$

Since we will be interested in the solutions to this equation $\gamma \ll 1$, the $O(\gamma^3)$ contributions can be neglected giving

$$\beta\gamma^2+\alpha\gamma+4\xi(d+\xi)-\frac{24\xi^2(d-1)}{d}=0, \quad (38)$$

where $\alpha=(d+4\xi)(d+2)+4\xi(\xi-1)-48\xi^2(d^2-1)/d^2(d+2)$ and $\beta=2d+6-1/d(d+2)$. We can see that if

$$4\xi(d+\xi)>24\frac{\xi^2(d-1)}{d}, \quad (39)$$

this equation has two negative roots. In this case the correlation function is dominated by the solution r^{γ_1} with $\gamma_1>\gamma_2$ and the $O(r^{-\gamma_1})$ term disappears in the limit $r\to 0$. The anomalous scaling is dominated by zero modes. However, if

$$4\xi(d+\xi)<24\frac{\xi^2(d-1)}{d} \quad (40)$$

the homogeneous equation produces two roots $\gamma_1>0$ and $\gamma_2<0$ and the scaling of the correlation function $G\propto r^0$ is forced by the $O(r^{-\gamma})$ contribution to (37). The amplitude of the correlation function is easily determined from (37). This result means that the zero mode disappears and the scaling of the dissipation rate correlation function is ‘‘normal’’ as in the Kolmogorov theory of turbulence. The critical value of the ratio $z=\xi/d$ at which the crossover takes place is

$$z_c \approx \frac{d}{5d-6}. \quad (41)$$

We can see that $1/5<z_c<1/2$. At large d the crossover parameter $z_c\to 1/5$ which reasonably justifies the approximation $z\ll 1$ used in the derivation. Formula (41) depends on the numerical values of the coefficients in the equations of motion calculated in the range of a parameter variation where the tensor γ_{ijkl} can be considered a potential tensor. That is why, in principle, it is dangerous to use it in the interval $\xi/d=O(1)$. Indeed, this result is inconsistent with the systematic expansion in powers of ξ/d developed above, since the $O(1/d)$ potentiality-violating correction to (31) was neglected in deriving (41). That is why the disappearence of the zero mode given by (41) can be considered only as a possible scenario of what can happen in the range $\xi/d=O(1)$. Relation (41), however, can be approximately correct if the neglected terms in (31) are numerically small. In this case the corrections, neglected in (31) are small and cannot violate the balance leading to the disappearence of the zero mode at $\xi/d\approx 1/5$. Another important conclusion of this work is that the fourth-order moment of the scalar difference is completely described by the second-order moment $S_2(r)$ and the dissipation rate correlation function $G(r)$. This is similar to the semiheuristic $K-\mathcal{E}$ model, widely used in engineering for quantitative description of turbulent flows. It is interesting that this model, leading to the anomalous scaling of S_4 can be derived from the equation for a passive scalar advected by a rapidly changing in time random velocity field in the limit $\xi/d\to 0$.

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[1] R.H. Kraichnan, Phys. Fluids **11**, 945 (1968).
[2] M. Vergassola, Phys. Rev. E **53**, 3021 (1996).
[3] V. Borue and V. Yakhot, Phys. Rev. E **53**, 5576 (1996).
[4] A. Gawedzki and A. Kupianen, Phys. Rev. Lett. **75**, 3608 (1995).
[5] M. Chertkov, G. Falkovich, I. Kolokolov, and V. Lebedev, Phys. Rev. E **52**, 4924 (1995).
[6] A.S. Monin and A.M. Yaglom, *Statistical Fluid Mechanics* (MIT Press, Cambridge, MA, 1975), Vol. 2.